Modelling the Temperature Changes of a Hot Plate and Water in a Proportional, Integral, Derivative Control System

Grant Hutchins

1. Introduction

Temperature changes in a hot plate heating system can be modelled with a set of differential equations. These equations and data gathered from an actual Proportional, Integral, Derivative (PID) temperature controller setup are modelled in a way that predicts the future performance of this controller type.

The system modelled has two temperature sensors: one on the heating coil and another in a cup of water on the coil. As power enters the system through the coil, the coil heats up. Some of this heat is drawn away by surrounding air and by the water in the cup. The water then gains heat from the coil and loses some heat to the air. The power level of the heater is then controlled by a PID system that analyzes the difference between temperature of the water in the cup and some predetermined set value.

Two thermocouples are used to measure the temperatures of the hot plate and of the water. The voltages generated by these thermocouples are then amplified by a differential amplifier circuit and measured through an analog-to-digital converter onto a laptop setup. An algorithm will then produce an output that drives a relay that is wired into the power cord of the hotplate, rapidly switching it on and off using pulse width modulation (PWM). The power input to the hot plate will be optimized so that it keeps the temperature of the water at the set value.

The system is modelled with differential equations both in MATLAB and in Simulink and data is collected with MATLAB and its Digital Acquisition Toolbox.

2. Procedure

(a) *PID Control*

Proportional, Integral, and Derivative control is a powerful control scheme that allows for a good amount of tweaking. PID control works by analyzing three properties of the difference between the set value and the actual value, or error.

First the Proportional control takes the value of the error and multiplies it by some gain to create a proportional effort. Next the Integral control keeps integrating the value and multiplies this amount by some other gain. Finally the Derivative controller analyzes the derivative of the error and multiplies this by a third gain.

The three effort values are combined to form the output of the PID control mechanism. The Proportional portion of the controller tends to affect the rate at which the actual value approaches the set value, the Integral portion tends to ensure that the final value as time increases tends more closely to the actual set value, and the Derivative portion tends to affect the amount of delay in the system, effectively changing the time constant τ of the actual value curve.

(*b*) *Modelling Heat Transfer*

This model assumes that the hot plate heats and cools such that Newton's Law of Heating should hold. Applying Newton's Law to model the change in hot plate coil temperature gives:

$$\frac{dT_c}{dt} = P_{in} + k_{ac}\left(T_a - \frac{dT_c}{dt}\right) + k_{wc}\left(\frac{dT_w}{dt} - \frac{dT_c}{dt}\right), \tag{2.1}$$

where P_{in} is the power input to the hot plate, T_c is the temperature of the hot plate coil, T_a is the local air temperature, and T_w is the temperature of the water.

Usually Newton's Law would not include a power term. This has been added to account for the fact that electricity is heating the hot plate coil. k_{wc} is a gain constant which accounts for how the relationship between the thermal masses of the water and the hot plate coil affects the transfer of heat from the water to the coil. Similarly, k_{ac} accounts for the transfer of heat from the air to the hot plate coil.

By the same process, the model for the temperature of the water is

$$\frac{dT_w}{dt} = k_{cw}\left(\frac{dT_c}{dt} - \frac{dT_w}{dt}\right) + k_{aw}\left(T_a - \frac{dT_w}{dt}\right), \tag{2.2}$$

where k_{cw} and k_{aw} are the gain constants associated with how the coil heats the water and how the air heats the water, respectively.

(*c*) *Constant Selection*

The k values are related to the time it takes for the temperature to change by a factor of e^{-1}. This amount of time is usually represented by τ. The values of k found are proportional to the inverse of the τ for each pair of bodies exchanging heat in the experiment.

Approximations of the appropriate P_{in} and k values were found by trial and error and adjusted based on the properties of the original curves. These values are hard-coded into the model along with initial conditions for the three temperatures. Each initial temperature is 20 degrees Celsius.

(*d*) *Generating Data*

The final model is a numerical approximation of the solution of a system of differential equations. This procedure solves differential equations by estimating new points based on the derivatives of given points. The step size of the independent variable is selected by the algorithm (ODE45 in MATLAB) to ensure that the error stays within a certain set value, in this case 1×10^{-6}.

The value of the output effort is limited to between 0 and 3.5. The hot plate cannot take a negative input power, and the constant 3.5 is a calibration figure for the amount of power observed when the hot plate is left on full power.

Another version of the model is prepared as a systems diagram. This model sets up the same systems of differential equations but has a graphical representation . This model is also solved by a differential equations system numerical approximation solver.

The output effort is limited by a Saturation control. The saturation control limits the value to between 0 and 3.5, just as in the previous version of the model.

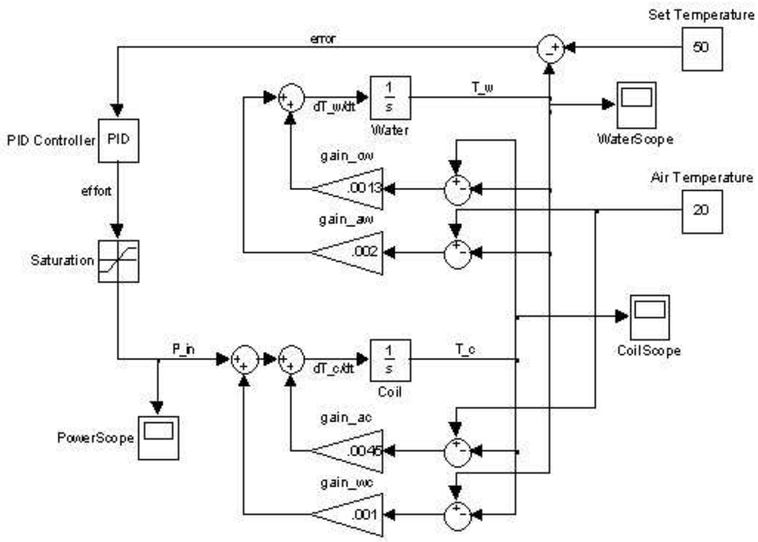

Figure 1. Systems Diagram for PID-controlled Hot Plate Temperature System

This diagram represents the relationship between a function and its derivative through the use of an integrator. The two integrators in Figure 1 are the boxes marked "$\frac{1}{s}$". The input of each integrator is the derivative of what is output. These output values in this model are the temperatures of the water and of the coil. These values, as well as the set temperature and air temperature (both constants) are fed back into the derivatives after passing through various summations that perform addition and subtraction and gains that perform multiplication. The paths in which these values travel and over which they change represent the differential equations for modelling the heat transfer.

(e) *Model Results*

Figures 2 through 5 show the results of the simulations.

(f) *Data Collection*

Nicole Hori and I collected data to test the accuracy of this model. Two thermocouples measure the temperatures of both the hot plate coil and the water. This generates a voltage which is amplified through a differential amplifier with a gain of 100. This is algorithmically converted into a Celsius temperature measurement.

The algorithm analyzes the value, running integral, and instantaneous derivative of the water temperature and processes it through a PID algorithm as mentioned above. The gains used are shown in Table 1.

The experiment gathered data for 30 minutes. The temperature data for the water and the hot plate coil are recorded in Figure 6.

3. Analysis

The goal of the model was to assist in predicting how the experiment would run and which PID control gain constants to use. The results show that the actual experiment controlled the temperature more quickly than the model predicted. This is most likely because of inaccuracies in the model relating to how quickly the water cools.

The temperature of the water does not overshoot the set value in the experiment, whereas there is a considerable amount of overshoot in the model. This overshoot is somewhat desirable because it shows that a system is critically damped. The experiment results appear to be underdamped. Perhaps more desirable values of the PID gains could generate a critically damped scenario that converges near the set temperature at a faster rate.

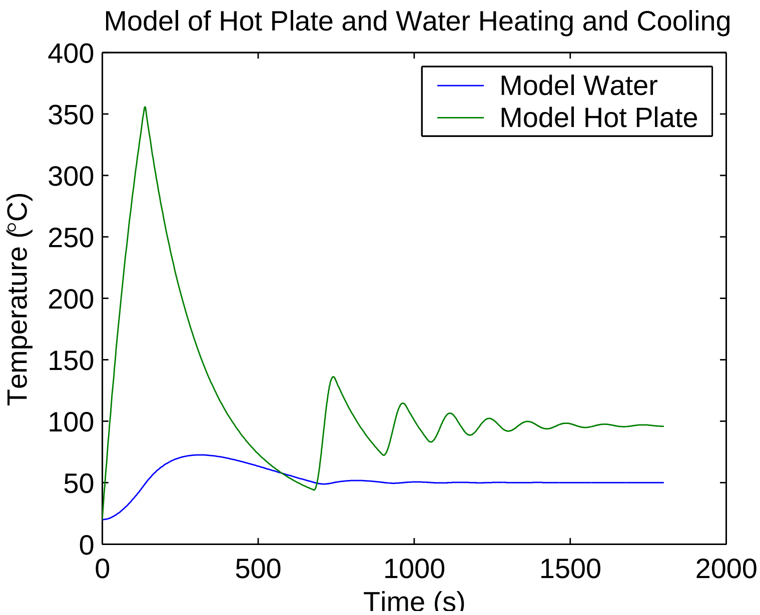

Figure 2. Numerical Model Results

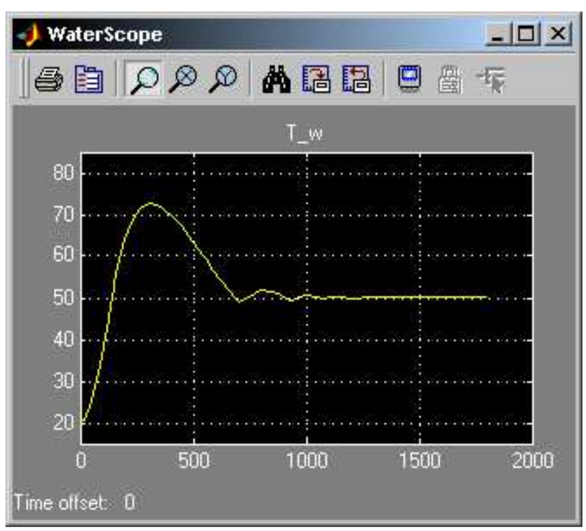

Figure 3. Systems Diagram Model for Water Temperature

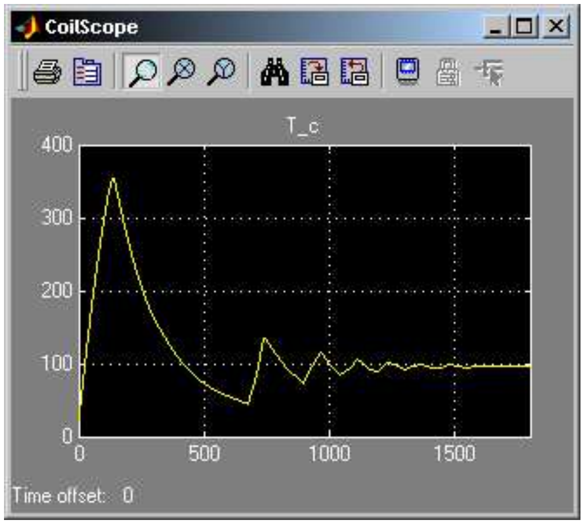

Figure 4. Systems Diagram Model for Hot Plate Coil Temperature

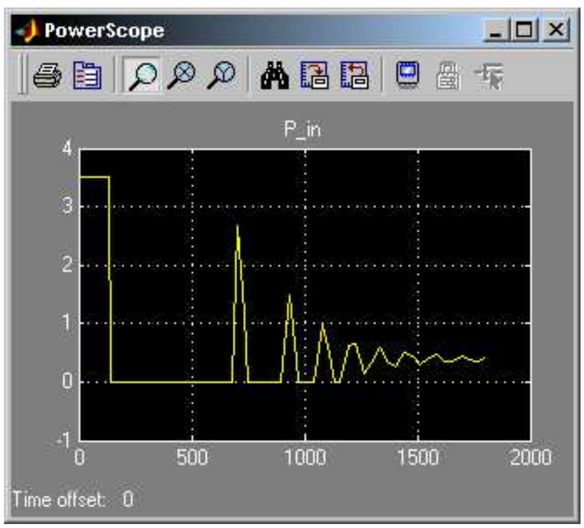

Figure 5. Systems Diagram Model for Power Input to Hot Plate

Proportional	2
Integral	-.00012
Derivative	0

Table 1. *PID Control gains*

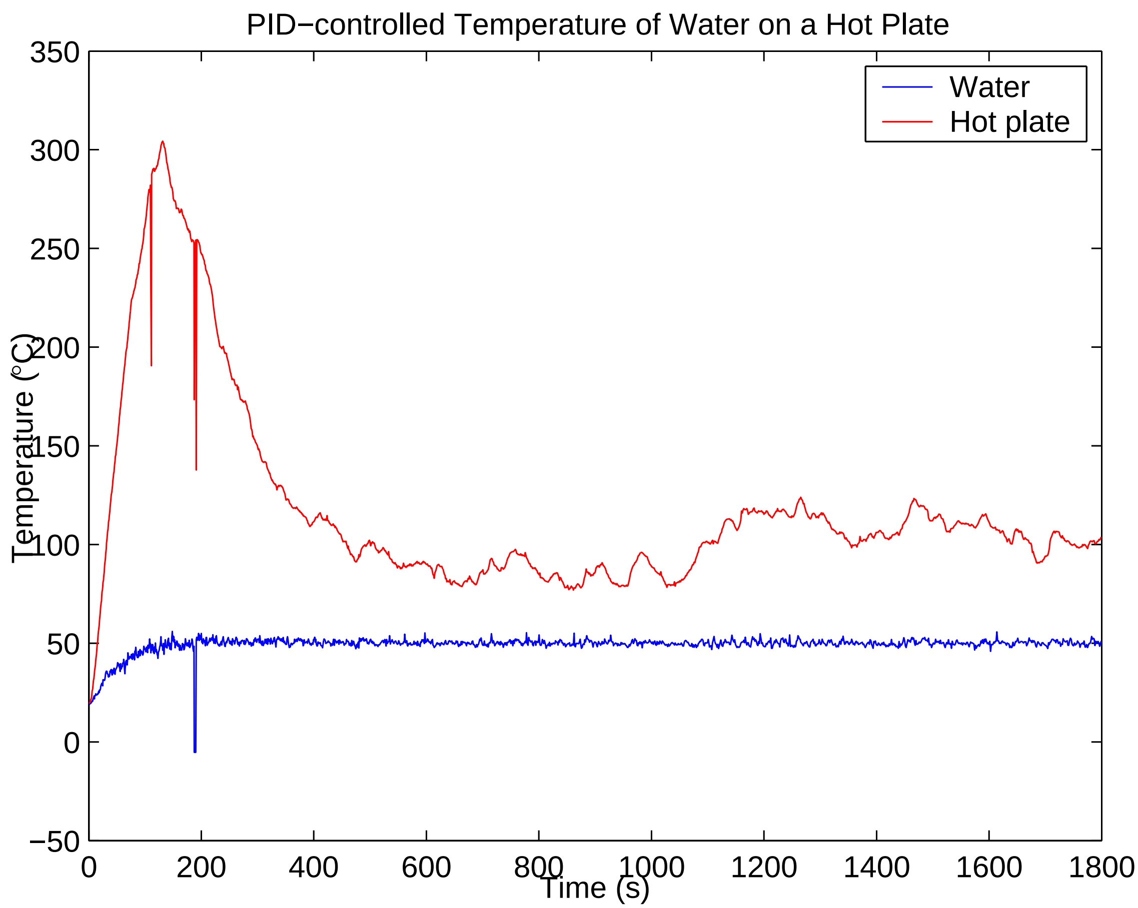

Figure 6. Experiment Results